USA Mathematical Talent Search

PROBLEMS / SOLUTIONS / COMMENTS
Round 2 - Year 12 - Academic Year 2000-2001

Gene A. Berg, Editor

1/2/12. A well-known test for divisibility by 19 is as follows: Remove the last digit of the number, add twice that digit to the truncated number, and keep repeating this procedure until a number less than 20 is obtained. Then, the original number is divisible by 19 if and only if the final number is 19. The method is exemplified on the right; it is easy to check that indeed 67944 is divisible by 19, while 44976 is not.

Find and prove a similar test for divisibility by 29.

$$
\begin{array}{cccccc}
 & 6 & 7 & 9 & 4 & \cancel{4} \\
 & & & & 8 & \\
\hline
 & 6 & 8 & 0 & \cancel{2} & \\
 & & & 4 & & \\
\hline
 & 6 & 8 & \cancel{4} & & \\
 & & 8 & & & \\
\hline
 & 7 & \cancel{6} & & & \\
1 & 2 & & & & \\
\hline
1 & 9 & & & &
\end{array}
\qquad
\begin{array}{cccccc}
 & 4 & 4 & 9 & 7 & \cancel{6} \\
 & & & 1 & 2 & \\
\hline
 & 4 & 5 & 0 & \cancel{9} & \\
 & & 1 & 8 & & \\
\hline
 & 4 & 6 & \cancel{8} & & \\
 & 1 & 6 & & & \\
\hline
 & 6 & \cancel{2} & & & \\
 & 4 & & & & \\
\hline
1 & 0 & & & &
\end{array}
$$

Solution 1 by Nina Boyarchenko (10/PA):
Method: Remove the last digit of the number and add three times that digit to the truncated number. Keep repeating this procedure until a number less than 30 is obtained. Then the original number is divisible by 29 if and only if the final number is 29.

Proof: The given number can be represented as $10a + b$, where b is the last digit of the number and a is the number when b is removed. When b is removed and when $3b$ is added to the truncated number, it becomes $a + 3b$.

$10a + b$ is divisible by 29 if and only if $10a + b + 29b$ is divisible by 29.
$10a + b + 29b = 10a + 30b = 10(a + 3b)$ is divisible by 29 if and only if $a + 3b$ is divisible by 29 as 10 is not divisible by 29 and 29 is prime. Also, since 29 is the only number between 0 and 30 that is divisible by 29, the original number $10a + b$ is divisible by 29 if and only if the final number is divisible by 29.

Solution 2 by Benjamin Armbruster (12/AZ): The test for divisibility by 29 is as follows: Remove the last digit of the number (x), add three times that digit to the truncated number (obtaining x'), and keep repeating this procedure until a final number less than 30 is obtained. Then, the original number is divisible by 29 if and only if the final number is 29.

Proof. Let the original number be x. The last digit of x is $x(\text{mod } 10)$. After removing the last digit one has

$$\frac{x-(x(\text{mod }10))}{10}$$

If one then adds three times that last digit one obtains

$$x' = \frac{x-(x(\text{mod }10))}{10} + 3(x(\text{mod }10)) = \frac{x+29(x(\text{mod }10))}{10}$$

Hence,

$$x' \equiv ([(x+29)(x(\text{mod }10))]/10)(\text{mod }29)$$
$$10x' \equiv (x+29)(x(\text{mod }10))(\text{mod }29)$$
$$10x' \equiv x(\text{mod }29)\,.$$

Since 29 is prime, this means that $x' \equiv 0$ if and only if $x \equiv 0$. This means that the final number is divisible by 29 if and only if the original number was. Because $x' = \frac{(x+29)(x(\text{mod }10))}{10}$, x' will always be less than x for positive x. This means that if you repeat the procedure long enough, you will get a number less than 30. Then, any number less than 30 is divisible by 29 if and only if it is 29.

Solution 3 by Sean Markan (11/MA):
A similar test for divisibility by 29 is to remove the last digit of the number, add three times that digit to the truncated number, and repeat the process until the resulting number is less than 30. The original number is divisible by 29 iff the final number is 29.

To prove this test, we first show that $10a+b$ is a multiple of 29 if and only if $a+3b$ is a multiple of 29:

$$a+3b \equiv 0(\text{mod }29)$$
$$\Leftrightarrow 10a+30b \equiv 0(\text{mod }29)$$
$$\Leftrightarrow 10a+b \equiv 0(\text{mod }29)$$

Now, starting with a number *ab*, where *a* is an integer, *b* is a single digit, and *ab* represents $10a+b$, tripling the last digit and adding it to the truncated number yields $a+3b$, which is a multiple of 29 if and only if the previous number is. If $10a+b>29$, then $a+3b$ is a smaller number also. So, by repeating this process we arrive at a number less than 30 which is a multiple of 29 iff the original number was. Therefore, if the final number is 29, then the original number was divisible by 29; otherwise the original number was not divisible by 29.

Extension by Laura Pruitt (11/MA): An essentially identical rule will work for any number one less than a multiple of ten: for 39, multiply by 4; for 49, by 5; etc. This format is also the basis of the more common divisibility rule for 9, which simply adds up the digits to get a multiple of 9. This is the essential difference: by removing all available multiples of the number at each step (by “bypassing” all tens), our rule gives the number itself as the final answer rather than any multiple of that number.

Editor’s Comment: We thank our Problem Editor, Dr. George Berzsenyi, for this problem.

2/2/12. Compute $1776^{1492!} \pmod{2000}$; i.e., the remainder when $1776^{1492!}$ is divided by 2000. (As usual, the exclamation point denotes factorial.)

Solution 1 by Jason Chiu (12/NY): Answer: 1376.

Powers of 1776 (mod 2000)

1776^1 has remainder 1776,

1776^2 has remainder 176,

1776^3 has remainder 576,

1776^4 has remainder 976,

1776^5 has remainder 1376,

1776^6 has remainder 1776,

1776^7 has remainder 176,

and so on.

Since $1776^6 \equiv 1776^1 \equiv 1776 \pmod{2000}$, $1776^n \equiv 1776^{n-5} \pmod{2000}$ for all $n > 5$ and we may consider the exponent (mod 5). It is plain that 1492! is divisible by 5, so that

$1776^{1492!} \equiv 1776^5 \equiv 1376 \pmod{2000}$

Therefore, $1776^{1492!}/2000$ has remainder **1376**.

Solution 2 by Eugene Fridman (12/IL): Answer: **1376**.

We begin by introducing a lemma.

Lemma: For all positive integers n, $1376^n \equiv 1376 \pmod{2000}$.

Proof of Lemma: We prove the lemma by induction. We first notice that $1376^1 \equiv 1376 \pmod{2000}$ and $1376^2 = 1893376 \equiv 1376 \pmod{2000}$. For our induction hypothesis, we assume the statement is true for $n = k$, i.e. that $1376^k \equiv 1376 \pmod{2000}$. We then prove that the statement is true for $n = k+1$. That is, we prove that $1376^{k+1} \equiv 1376 \pmod{2000}$. To do so, we start with the inductive hypothesis and multiply both sides of the congruence equation by 1376 to obtain $1376 \cdot 1376^k \equiv 1376^2 \pmod{2000}$, or $1376^{k+1} \equiv 1376^2 \pmod{2000}$. Since $1376^2 \equiv 1376^1 \pmod{2000}$, the congruence above can be written as $1376^{k+1} \equiv 1376 \pmod{2000}$, and our proof is complete.

We now compute the required quantity by noticing that $1776^5 \equiv 1376 \pmod{2000}$. Since 5 divides 1492! we can say that

$$1776^{1492!} \equiv (1776^5)^{\frac{1492!}{5}} \equiv 1376^{\frac{1492!}{5}} \pmod{2000}$$

According to the lemma above,

$$1376^{\frac{1492!}{5}} \equiv 1376 \pmod{2000}$$

so

$$1776^{1492!} \equiv 1376 \pmod{2000}.$$

Hence our answer is **1376**.

Solution 3 by Charles Wang (12/IL):

Since $2000 = 16 \cdot 125$, we can look at this number mod 125 and mod 16 and use the Chinese Remainder Theorem to find this number mod 2000. First, $1776^{1492!} \equiv 0 \pmod{16}$ since 1776 is a multiple of 16 so any power of 1776 is a multiple of 16 also. It is also well known that $x^{\phi(n)} \equiv 1 (\mathrm{mod} n)$ where $(n, x) = 1$ and $\phi(n)$ is the totient function, a function that counts the number of positive integers less than and relatively prime to n. Since $\phi(125)$ is less than 125, which in turn is less than1492, we know that $\phi(125) | 1492! \quad \Rightarrow 1492! = \phi(125) \cdot k$ where $k \in Z$.

Since $(125, 1776) = 1$, $1776^{\phi(125)} \equiv 1 \pmod{125} \quad \Rightarrow 1776^{\phi(125) \cdot k} \equiv 1^k \equiv 1 \pmod{125}$. Therefore $1776^{1492!} \equiv 1776^{\phi(125) \cdot k} \equiv 1 \pmod{125}$.

Using the Chinese Remainder Theorem$1776^{1492!} \equiv 0 \cdot 125 \cdot a + 1 \cdot 16 \cdot b \pmod{16 \cdot 125}$, where $125 \cdot a \equiv 1 \pmod 8$ and $16 \cdot b \equiv 1 \pmod{125}$. Since $125 \cdot a$ is multiplied by 0, we need only solve for b in this equation. Running quickly through the equations we find that $16 \cdot 86 \equiv 1 \pmod{125}$. Plugging back in, we find that

$1776^{1492!} \equiv 16 \cdot 86 \pmod{2000} \quad \Rightarrow \quad 1776^{1492!} \equiv 1376 \pmod{2000}$.

Solution 4 by Anatoly Preygel (10/MD):

Euler's Extension to Fermat's Little Theorem states: If $(a,m) = 1$, then $a^{\phi(m)} \equiv 1 (\mathrm{mod} m)$.

So we have $a^{100} \equiv 1 \pmod{125}$ for all a with gcd(a, 125) = 1.

We see that $16 \cdot 125 = 2000$, and 16 divides 1776 so $1776^{1492!} \equiv 0 \pmod{16}$.

Let us consider $1776^{1492!} \pmod{125}$. By Euler's theorem above, since gcd(125, 1776) = 1, and since 5|1492!:

$$1776^{1492!} \equiv 1 \pmod{125}.$$

We now wish to solve for the system of congruences $n \equiv 1 \pmod{125}$, $n \equiv 0 \pmod{16}$, which we can do by simple trial. Of all the numbers between 0 and 2000 congruent to 1 mod 125, namely

251, 376, 501, 626, 751, 876, 1001, 1251, 1376, 1501, 1626, 1751, 1876, only 1376 is divisible by 16.

Thus we see that $1776^{1492!} \equiv 1376 \pmod{2000}$

Editor's Comment: We are thankful to Dr. Peter Anspach of NSA for this nice problem. Remarkably, $1492^{1776!} \pmod{2000} = 1376$ as well, but seems harder to prove. For a brief introduction to *Euler's totient function* $\phi(m)$, *Euler's generalization of Fermat's Theorem*, and *the Chinese Remainder Theorem* see the Solutions to Round 1 of Year 12 on these web pages.

3/2/12. Given the arithmetic progression of integers

$$308, 973, 1638, 2303, 2968, 3633, 4298,$$

determine the unique geometric progression of integers,

$$b_1, b_2, b_3, b_4, b_5, b_6$$

so that

$$308 < b_1 < 973 < b_2 < 1638 < b_3 < 2303 < b_4 < 2968 < b_5 < 3633 < b_6 < 4298.$$

Solution by Rishi Gupta (8/CA): First I tried to find a number x such that $b_1 \cdot x = b_2$, $b_2 \cdot x = b_3$, $b_3 \cdot x = b_4$, etc.

I started by finding the smallest value x could equal. Since b_6 and b_1 are 5 sequence numbers apart, $b_6 = b_1 \cdot x^5$. I took the largest value for b_1, which is 972, and the smallest value for b_6, which is 3634. Therefore, $3634 \le 972x^5$ and $1.301 \le x$.

Then I found the maximum value for x, using the smallest b_4 and the largest b_6, so $4297 \ge 2304x^2$, and $x \le 1.37$.

Therefore $1.301 \le x \le 1.37$.

Now since $b_1 \cdot x^5 = b_6$ and the sequence is integral, when x is written as a fraction b_1 must be divisible by the denominator to the fifth power. Any fraction with a denominator greater than 3 will not work, because $4^5 = 1024$ and b_1 must be less than 973. Neither 1/1 nor 1/2 have multiples between 1.30 and 1.37, so that leaves us with a denominator of 3. Since 4/3 is the only multiple of 1/3 that is between 1.30 and 1.37, $x = 4/3$.

Since $3^5 = 243$, b_1 is a multiple of 243. Because b_1 is between 309 and 972, it can only be 486,

729, or 972. Also, since $729 \cdot (4/3)$ is not bigger than 973 as required for b_2, 486 and 729 are ruled out. That leaves b_1 equal to 972, $b_2 = 972 \cdot (4/3) = 1296$,
$b_3 = 1296 \cdot (4/3) = 1728$, $b_4 = 1728 \cdot (4/3) = 2304$, $b_5 = 2304 \cdot (4/3) = 3072$, and $b_6 = 3072 \cdot (4/3) = 4096$.

Therefore the solution is

$b_1 = 972$
$b_2 = 1296$
$b_3 = 1728$
$b_4 = 2304$
$b_5 = 3072$
$b_6 = 4096$

Editor's Comment: This problem was inspired by Problem 7 of the Second Selection Examination held in Bucharest, on April 25, 1999. We are indebted to Károly Dáné of Romania for calling this problem to our attention.

4/2/12. Prove that every polyhedron has two vertices at which the same number of edges meet.

Solution 1 by Lisa Fukui (12): At least three edges must meet at every vertex of a polyhedron.

If a polyhedron had *n* vertices and every vertex had different numbers of edges meeting, then the number of edges meeting at the vertex with the most edges would be at least $n + 2$.

A polyhedron with *n* vertices cannot have a vertex with more than $n - 1$ edges meeting, since each edge is a segment between two vertices.

Therefore, there are no polyhedra that have a different number of edges meeting at every vertex. So every polyhedron has two vertices at which the same number of edges meet.

Solution 2 by Agustya Mehta (9/OH): Let the polyhedron have *n* vertices.

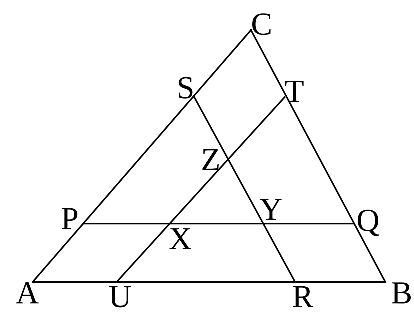
The minimum number of edges that can meet at a vertex is 3. The maximum number of edges that can meet at a vertex is $n - 1$. Let us assume that we have a pigeon sitting on each vertex of our polyhedron (we assume that these pigeons can count and read), and we have pigeonholes marked 3, 4, 5, ..., $n - 1$. The pigeons count the number of edges that meet at their vertex, and fly to the pigeonhole with the same number. Since we have n pigeons and only $n - 3$ pigeonholes, there must be at least one pigeonhole with more than one occupant. The vertices from where these pigeon roommates originally flew, have the same number of edges meeting. Thus there are at least two vertices that have the same number of edges meeting. (The argument is still valid even if we do not have pigeons that can count and read <smile>!)

Solution 3 by Sofia Leibman (8/OH): In a polyhedron with n vertices, the number of edges that can meet at one vertex is an integer between 3 and $n-1$. But it is impossible to choose n different numbers from $n-3$ numbers (3, 4, 5, ..., $n-1$). So there must be at least two vertices where the same number of edges meet.

Editor'scomment: This problem parallels Problem M15, which appeared in the September/ October 1990 issue of *Quantum*. It demonstrates the importance of the extreme case in problem solving.

5/2/12. In ΔABC, segments PQ, RS, and TU are parallel to sides AB, BC, and CA, respectively, and intersect at the points X, Y, and Z, as shown in the figure on the right.

Determine the area of ΔABC if each of the segments PQ, RS, and TU bisects (halves) the area of ΔABC, and if the area of ΔXYZ is one unit. Your answer should be in the form $a + b\sqrt{2}$, where a and b are positive integers.



Solution by Rachel Johnson (11/MN): The area of triangle ABC can be determined using ratios. Let AB be x. When a triangle's area is cut in half by a line parallel to the base, the ratio of the base of the original to the new base is $\sqrt{2}$:1. So PQ is $(x\sqrt{2})/2$. Since triangles PCQ, UTB, and ASR have the same areas (half the total), and have the same angles, they are congruent. It follows that PX and YQ equal $(2x - x\sqrt{2})/2$. Subtracting PS and YQ from PQ shows that XY equals $(3x\sqrt{2} - 4x)/2$. From this, the ratio of the base of ABC to that of XYZ is $2/(3\sqrt{2} - 4)$:1. To get the ratio of the area of ABC to that of XYZ, the ratio of bases is squared: $2/(17 - 12\sqrt{2})$:1. Since the area of XYZ is 1, the area of ABC is $2/(17 - 12\sqrt{2})$, or in $a + b\sqrt{2}$ form, $34 + 24\sqrt{2}$.

Editor's comments: For more information on the area bisectors of a triangle, the reader is referred to the article "Halving the Triangle" by J. A. Dunn and J. E. Pretty in Number 396 (May 1972) of *The Mathematical Gazette*. We thank our problem editor, George Berzsenyi for posing this problem.